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Solving a Special Fractional Integral

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***Abstract*: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study a special fractional integral. The exact solution of this fractional integral can be obtained by using some techniques. In addition, our result is a generalization of the result of traditional calculus.**

***Keywords:* Jumarie type of R-L fractional calculus, New multiplication, Fractional analytic functions, Fractional integral.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis which deals with the research and applications of integrals and derivatives of arbitrary order. In recent decades, the field of fractional calculus has attracted the interest of researchers in diverse scientific fields such as mechanics, physics, electrical engineering, viscoelasticity, economics, bioengineering, and control theory [1-11]. However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following special α-fractional integral:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\,[p+q\cos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right], \tag{1}$$

where $0<\alpha\leq 1$, and p,q are real numbers. Using some techniques, the exact solution of this α-fractional integral can be obtained. In fact, our result is a generalization of classical calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{3}$$

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where $\Gamma(\ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([18])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{5}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{7}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ & = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ & = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{9}$$

Definition 2.5 ([21])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha n}, \tag{12}$$

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and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{13}$$

Definition 2.6 ([22])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Definition 2.7 ([23])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha (-1)}$.

III. MAIN RESULT AND EXAMPLES

In this section, we find the exact solution of a special fractional integral. On the other hand, we propose some examples to illustrate our result.

Theorem 3.1: *Suppose that* $0 < \alpha \leq 1$, *and* p, q *are real numbers.*

Case 1. If $p^2 > q^2$, *then*

$$\left({}_0I_x^\alpha\right)\left[\,[p + q\,cos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right] = \frac{2}{p-q}\cdot\sqrt{\frac{p-q}{p+q}}\cdot arctan_\alpha\left(\sqrt{\frac{p-q}{p+q}}\,tan_\alpha\left(\frac{1}{2}x^\alpha\right)\right). \tag{17}$$

Case 2. If $p^2 < q^2$, *then*

$$\left({}_0I_x^\alpha\right)\left[\,[p + q\,cos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right] = \frac{1}{q-p}\cdot\sqrt{\frac{q-p}{q+p}}\cdot Ln_\alpha\left(\left|\left[tan_\alpha\left(\frac{1}{2}x^\alpha\right) + \sqrt{\frac{q+p}{q-p}}\right]\otimes_\alpha\left[tan_\alpha\left(\frac{1}{2}x^\alpha\right) - \sqrt{\frac{q+p}{q-p}}\right]^{\otimes_\alpha(-1)}\right|\right). \tag{18}$$

Proof *Case 1.* If $p^2 > q^2$, then

$$\left({}_0I_x^\alpha\right)\left[\,[p + q\,cos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[p\left(\left[cos_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2} + \left[sin_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2}\right) + q\left(\left[cos_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2} - \left[sin_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2}\right)\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[(p+q)\left[cos_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2} + (p-q)\left[sin_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[(p+q) + (p-q)\left[tan_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left[sec_\alpha\left(\frac{1}{2}x^\alpha\right)\right]^{\otimes_\alpha 2}\right]$$

$$= \frac{2}{p-q} \cdot \left({}_0 I_x^\alpha \right) \left[\left[\frac{p+q}{p-q} + \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{x_0} D_x^\alpha \right) \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right] \right]$$

$$= \frac{2}{p-q} \cdot \sqrt{\frac{p-q}{p+q}} \cdot arctan_\alpha \left(\sqrt{\frac{p-q}{p+q}} tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right).$$

Case 2. If $p^2 < q^2$, then

$$\left({}_0 I_x^\alpha \right) \left[[p + q cos_\alpha (x^\alpha)]^{\otimes_\alpha (-1)} \right]$$

$$= \left({}_0 I_x^\alpha \right) \left[\left[(q+p) - (q-p) \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left[sec_\alpha \left(\frac{1}{2} x^\alpha \right) \right]^{\otimes_\alpha 2} \right]$$

$$= \frac{2}{q-p} \cdot \left({}_0 I_x^\alpha \right) \left[\left[\frac{q+p}{q-p} - \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right]^{\otimes_\alpha 2} \right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_{x_0} D_x^\alpha \right) \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right] \right]$$

$$= \frac{2}{q-p} \cdot \frac{1}{2} \sqrt{\frac{q-p}{q+p}} Ln_\alpha \left(\left| \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) + \sqrt{\frac{q+p}{q-p}} \right] \otimes_\alpha \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) - \sqrt{\frac{q+p}{q-p}} \right]^{\otimes_\alpha (-1)} \right| \right)$$

$$= \frac{1}{q-p} \cdot \sqrt{\frac{q-p}{q+p}} \cdot Ln_\alpha \left(\left| \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) + \sqrt{\frac{q+p}{q-p}} \right] \otimes_\alpha \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) - \sqrt{\frac{q+p}{q-p}} \right]^{\otimes_\alpha (-1)} \right| \right).$$ Q.e.d.

Example 3.2: If $0 < \alpha \leq 1$, then

$$\left({}_0 I_x^\alpha \right) \left[[3 + 2cos_\alpha (x^\alpha)]^{\otimes_\alpha (-1)} \right] = \frac{2}{\sqrt{5}} \cdot arctan_\alpha \left(\frac{1}{\sqrt{5}} tan_\alpha \left(\frac{1}{2} x^\alpha \right) \right). \tag{19}$$

And

$$\left({}_0 I_x^\alpha \right) \left[[2 - 4cos_\alpha (x^\alpha)]^{\otimes_\alpha (-1)} \right] = -\frac{\sqrt{3}}{6} \cdot Ln_\alpha \left(\left| \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) + \frac{1}{\sqrt{3}} \right] \otimes_\alpha \left[tan_\alpha \left(\frac{1}{2} x^\alpha \right) - \frac{1}{\sqrt{3}} \right]^{\otimes_\alpha (-1)} \right| \right). \tag{20}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study a special fractional integral. By some techniques, we can obtain the exact solution of this fractional integral. Moreover, our result is a generalization of traditional calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

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